Spacetimes, electromagnetism and fluids (a revision of traditional concepts)*

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Abstract

In this talk r-form fields in spacetimes of any dimension D are considered ($r < D$). The weak-field Newtonian-type limit of Einstein's equations, in general, with relativistic sources is studied in the static case yielding a revision of the equivalence principle (intrinsically relativistic sources generate twice stronger gravitational fields and hyperrelativistic sources — *e.g.*, the stiff matter — generate four times stronger gravitational fields than non-relativistic sources). It is shown that analogues of electromagnetic field, strictly speaking, exist only in even-dimensional spacetimes. In (2+1)-dimensional spacetime, the field traditionally interpreted as "magnetic" turns out to be in fact a perfect fluid, and "electric", a perverse fluid (this latter concept arises inevitably in the r-form description of fluids for any D, and we consider here perverse fluids in (3+1)-dimensional spacetime too). New exact solutions of (2+1)-dimensional Einstein's equations with perfect and perverse fluids are obtained, and it is shown that in this case there exists a vast family of static solutions for non-coherent dust, in a sharp contrast to the (3+1)-dimensional case. New general interpretation of the cosmological term in D-dimensional Einstein's equations is given via the $(D-1)$-form field, and it is shown that this field is as well responsible (as this is the case in 3+1 dimensions) for rotation of perfect fluids [$(D-2)$-form fields], thus the "source" term in the corresponding field equations has to be interpreted as the rotation term.

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1 Introduction

We know — till now, mostly empirically — that our Universe is four-dimensional, with one temporal and three spatial dimensions. Naturally, there were some attempts to understand why this is really the case, but these attempts usually reduced to observation of how the "real" dimensionality affects the behaviour of particles and fields via the properties of dynamical equations, and testing strange peculiarities arising from application of the same equations to cases with other numbers of dimensions. However, in some versions of unified field theories, as well as in supersymmetry considerations, certain progress was made in understanding how our standard (and postulative) approach to the dimensionality of the Universe could follow from higher-dimensional theories (the compactification procedures also have to be mentioned in this connection). Still, there is no sign that any serious work was done in trying to trace an evolutionary formation of our 4-dimensional Universe from quantum theoretical and cosmological considerations, the formation which could begin with other number of dimensions, probably, with $1+1$ (see some remarks made in the last Section). In general, on the probable ways of formation of the laws of physics, a Chapter however was written by Wheeler on his "Pregeometry" [14], though it did not produce any response during so many years.

But, of course, there is not only dimensionality that matters; the very signature of spacetime, $+, -, -, -$ (in fact, already stated above when $3+1$ and $1+1$ were written), until recently remained to be an invariable and dumb postulate. Of course, there exists (in the literature) the twistor four-dimensional space, with its signature $+, +, -, -$, so well fitting many principles of special relativity and elementary particles theory. The twistor theory practically remains to be an isolated islet in the sea of the conventional four-dimensional $(+, -, -, -)$[1] theory, even without any attempts to formally apply to its geometry the mere classical scheme of general relativity. Probably, the most important breakthrough was achieved in studies of signature by H. van Dam and Y. Jack Ng [6]. Few months ago they have shown, using group-theoretical ideas of Wigner, that the non-trivial spin spectrum of particles in quantum theory is possible only in $3+1$ dimensions (more generally, in $n+1$ dimensions). See also the references given in [6].

In this talk I'll try to discuss some non-evolutionary backgrounds which

[1] Or, $(-, +, +, +)$, which is the same; see, however, interesting comments in [24].

could be of use in the realization of this ambitious but, presumably, inevitable program. After all, one has to clean the ground before starting to dig the hole and lay a foundation.

Greek indices will be used for spacetime D-dimensional coordinates while in the $(D-1)$-dimensional spatial sections we use Latin indices.

1.1 $D = 3 + 1$: well established theory

Almost everything we use today is based on the four-dimensional classical physics, even the quantum physics does not escape this destiny (we do not know the intrinsic language of the quantum world and have to apply the classical basic concepts which naturally are subjected there to the well known uncertainties). From our four-dimensional experience we know that one of the best studied and universally used fields, the (Maxwell) electromagnetic one, is (a) linear, (b) intrinsically relativistic even when it is time-independent, (c) there is a far-reaching analogy between its electric and magnetic parts, which we describe via the dual conjugation of the field tensor. In this talk I will show that these three properties are closely interrelated (see Appendices A and B).

The Maxwell field is the vector field (its potential is a 1-form). Thus it seems logical to study other r-form fields. The scalar (0-form) field was the first target for physicists, and the Klein–Gordon equation was derived already by Schrödinger as his first step towards his famous non-relativistic wave equation. The real mystery of the scalar field is why it serves mainly as a simple and nice example, but does not play any really fundamental and central rôle in today's physics (the scalar-tensor approach to gravity and the non-linear scalar field in general are clearly not directly relevant to the r-form fields study).

Strangely enough, only quite recently the 2-form field theory was directly applied to field theoretic description of perfect fluids [21, 22], interpreted as those via the automatically realized specific form of the stress-energy tensor,

$$T^{\mathrm{pf}} = (\mu + p) u \otimes u - p g. \tag{1.1}$$

The fact of so late understanding of the 2-form fields in 3+1 can be in a certain (indirect) sense related to the erroneous (but however only recent) denial by Weinberg [33] of the physical significance of 2- and 3-form fields (in particular, omitting the issue of radically different dynamical properties

of 0- and 2-form fields, see [22]). In this talk I fill a substantial gap in [21] where an alternative (to the perfect fluid) case in the 2-form field theory was not considered — the perverse fluid case, as I dare call it. Its inclusion is of importance at least since exactly this exotic case plays a significant rôle in 2+1-spacetime (then, of course, in the capacity of the 1-form field which describes there both perfect and perverse fluids).

Finally, the 3-form field (I call it the Machian field relating it also, somewhat arbitrarily, to the fundamental hypothetical field proposed by Sakurai [25]) was introduced in [21, 22]. In this talk the $(D-1)$-form field is considered which takes the place of the 3-form field in 3+1, playing the same rôle. This really exotic field, admitting either constant or arbitrary functions as solutions for its potential (thus having global and not local properties in a contrast to all other physical fields whose equations belong to the hyperbolic type), plays two very distinct rôles: (a) as a free field, it is responsible for existence or absence of the cosmological constant (in the latter case, and only then, the Machian field is intrinsically relativistic); (b) an interaction between this field and the 2-form field (in 3+1) is the only means to impart a rotation to the 2-form (fluid) field. Thus this 3-form field in 3+1-spacetime perfectly fits in Ernst Mach's world picture.

1.2 $D = 2 + 1$: hopes and prejudices

The three-dimensional (2+1) spacetime was repeatedly considered in many publications primarily to the end of finding guidelines of quantization of gravity, since the 2+1 case seemed to offer radical simplifications both in the canonical description of gravity and in the topological properties of the model spacetime (see, *e.g.*, [3, 10]). Naturally, the general attention was also attracted by the classical (2+1)-geometry, exact solutions of (2+1)-Einstein's equations, and in particular by the (2+1)-black holes: see [5] where Einstein's equations were reformulated (I prefer to use their traditional form below, without inclusion of the dimensionality in the gravitational constant); [4, 12, 1], with an important correction in [9]. This last correction proved to be only the beginning of a revision of the previous general approach based on some naïve prejudices concerning the electromagnetic fields in D dimensions, and the next (though not the last) step of this revision was [23].

Thus the vector field in 2+1 cannot be interpreted as electromagnetic field: it describes instead the (perfect and perverse) fluids. The case previously treated as electric field, is in fact the 2+1 perverse fluid, while the

2+1 "magnetic field" finds its physical interpretation as the perfect fluid (in [2] the authors came to a very similar, though not sufficiently general conclusion), exactly with the eigenvalues of its stress-energy tensor as the fluid energy density (corresponding to the eigenvector which is timelike in this latter case) and two equal pressures (isotropy) for any pair of spacelike vectors on the spacetime section orthogonal to the timelike eigenvector. Moreover, the inhomogeneity (in the dynamical field equations) whose presence was previously interpreted as electric charge and current distributions, now is proven to be *not a source term*, but the rotation term in the dynamical field equations; this conclusion is supported by several logically firm mathematical and physical arguments. This rotation term is due to the interaction between the 1-form and (Machian) 2-form fields in 2+1 spacetime, in a complete analogy with the situation in 3+1 where the ranks of the respective r-form fields have to be increased by one thus producing the room for the usual Maxwell (1-form in 3+1) field.

1.3 $D = n + 1$: the systematic approach

In this talk a systematic and self-consistent approach to electromagnetism and other (including model) fields is generalized to the $D = (n + 1)$-dimensional spacetimes. We shall use the concepts and notations introduced in Appendices A and B, in particular, concerning intrinsically relativistic and hyperrelativistic fields. The r-form fields ($r < D$), the respective field equations and stress-energy tensors are considered in Section 2 together with the eigenvalues of these T^{μ}_{ν}'s (with more details in other Sections dedicated to the specific fields), which make it easier not only to arrive at the physical interpretation of these fields, but also to provide adequate tetrad and vielbein bases indispensable in finding the corresponding exact solutions of Einstein's equations. The model fields (not seeming to be as fundamental as Maxwell's and Mach's fields) describing fluids are studied in Section 3 (nonrotating case) and further in Section 4 (Subsection 4.2, including rotation); not only the perfect fluids with all possible equations of state are considered, but also a new concept of the "perverse fluid" is introduced. In Section 4 it is shown that the free $(D - 1)$-form fields are equivalent to appearance of the cosmological term in Einstein's equations, the case of $\Lambda = 0$ being the intrinsically relativistic case of the $(D - 1)$-form fields (seeming to be as fundamental as the generalized Maxwell fields). Then, in Section 5, the electromagnetic (Maxwell-type) fields are generalized to all even-dimensional

spacetimes (in odd spacetime dimensions true electromagnetic-type fields are absent). Some new exact general relativistic solutions are reviewed and obtained for the perfect and perverse fluids in Section 6; in particular, a vast family of static non-coherent dust solutions in 2+1 is found, and it is shown that there are no rotating dust solutions in 2+1. Finally, in Section 7, the concluding remarks are given.

2 Skew rank r fields and their stress-energy tensors

The potential of a free r-form field is

$$A = \frac{1}{r!} A_{\alpha_1 \dots \alpha_r} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_r}, \tag{2.1}$$

a skew-symmetric tensor of rank r, and the field tensor (intensity) is

$$F = \frac{1}{(r+1)!} F_{\alpha_1 \dots \alpha_{r+1}} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_{r+1}} = dA, \tag{2.2}$$

while the only invariant on which the Lagrangian (density) depends, is

$$I = F_{\alpha_1 \dots \alpha_{r+1}} F^{\alpha_1 \dots \alpha_{r+1}}. \tag{2.3}$$

The free (but, in general, non-linear) r-form field equations then are

$$\left(\sqrt{|g|} \frac{dL}{dI} F^{\alpha_1 \dots \alpha_r \mu} \right)_{,\mu} = 0 \ \Leftrightarrow d \left(\frac{dL}{dI} * F \right) = 0. \tag{2.4}$$

The stress-energy tensor following from the Noether theorem (see [15, 16, 21]) takes the form

$$T_\alpha^\beta = -L\delta_\alpha^\beta + 2\frac{\partial L}{\partial g^{\alpha\mu}} g^{\beta\mu} = -L\delta_\alpha^\beta + 2(r+1)\frac{dL}{dI} F_{\alpha\mu_1 \dots \mu_r} F^{\beta\mu_1 \dots \mu_r}, \tag{2.5}$$

so that its trace reads

$$T_\alpha^\alpha = -DL + 2(r+1) I \frac{dL}{dI}. \tag{2.6}$$

Below, when this will be more convenient, we shall use other letters to denote the specific fields, their invariants, and the corresponding Lagrangians.

Since the concrete determination of eigenvalues and eigenvectors of the respective stress-energy tensors for arbitrary D and r crucially depends on these characteristics, we shall consider them for some concrete values of D and r only, and in the corresponding Sections. Here it is however worth mentioning that the Machian $(D-1)$-form field always has only one eigenvalue (equal to zero in the intrinsically relativistic case), and its eigenvector is completely arbitrary. The $(D-2)$-form field modeling perfect fluids has one timelike eigenvector corresponding to the single eigenvalue and $D-1$ spacelike eigenvectors with the same $(D-1)$-fold (degenerate) eigenvalue; these eigenvectors are orthogonal to the timelike one. The same type of field describing perverse fluids, has the same number of eigenvalues and linearly independent (D) eigenvectors, but the single eigenvalue corresponds to a spacelike eigenvector, and of the other $D-1$ eigenvectors orthogonal to it, one is timelike. The Maxwell-type field [existing only in even D, the potential being $(D/2-1)$-form] possesses two distinct eigenvalues, the both $(D/2)$-fold degenerate. Since the stress-energy tensor is real and symmetric, the system of its eigenvectors can be orthonormalized to form a natural basis in D dimensions (though for null fields, of course, this is not the case: there is always a pair of real null eigenvectors with mutual normalization, one of which determines the null direction in which the field is propagating: the specific property of the spacetimes whose signature is $+,-,\ldots,-$).

3 Fluids in field-theoretic description

3.1 Perfect fluids

Like in 3+1, the stress-energy tensor (2.5) of the $(D-2)$-form field in D dimensions reduces to

$$T^{\beta}_{\alpha} = 2I\frac{dL}{dI}b^{\beta}_{\alpha} - L\delta^{\beta}_{\alpha} \tag{3.1}$$

where

$$b^{\beta}_{\alpha} = \delta^{\beta}_{\alpha} - u_{\alpha}u^{\beta}, \quad b^{\beta}_{\alpha}u^{\alpha} = 0 = b^{\beta}_{\alpha}u_{\beta}, \quad u = \frac{f}{\sqrt{f\cdot f}}, \quad f = *F; \tag{3.2}$$

$*$ being the Hodge star (dual conjugation), while f is for perfect fluids a timelike covector, $f\cdot f > 0$, thus $u\cdot u = +1$. Since b^{β}_{α} is the projector on the (local) subspace orthogonal to the congruence of u, the latter is eigenvector

of the stress-energy tensor with the eigenvalue $(-L)$

$$T_\alpha^\beta u^\alpha = -Lu^\beta, \tag{3.3}$$

while any vector orthogonal to u is also eigenvector, now with the $(D-1)$-fold eigenvalue $2I\frac{dL}{dI} - L$. This is the property of the stress-energy tensor of a perfect fluid possessing the invariant mass density μ and pressure p (in its local rest frame):

$$\mu = -L, \quad p = L - 2I\frac{dL}{dI} \tag{3.4}$$

[the eigenvalue corresponding to the spacelike eigenvectors, is $(-p)$].

The free $(D-2)$-form field equations are

$$d\left(I^{1/2}\frac{dL}{dI}u\right) = 0. \tag{3.5}$$

Thus the free $r = D-2$ field case can describe only non-rotating fluids.

Perfect fluids characterized by the simplest equation of state

$$p = (2k-1)\mu, \tag{3.6}$$

correspond to the Lagrangian $L = -\sigma|I|^k$, $\sigma > 0$. In 3+1, the important special cases are: the incoherent dust ($p = 0$) for $k = 1/2$, intrinsically relativistic incoherent radiation ($p = \mu/3$) for $k = 2/3$, and hyperrelativistic stiff matter ($p = \mu$) for $k = 1$. There are two ways to consider the property of the fluid to be intrinsically relativistic and hyperrelativistic: (a) from the point of view of the relation between the temporal and spatial parts of the stress-energy tensor (essentially, the sign of its trace), this approach to be used below in this talk (see Table 1 in the Appendix B), or (b) taking into account the coefficients in the terms proportional to μ and p in (A.10), though in this case the (2+1)-spacetime obviously falls out of the consideration. In the approach (a) it is remarkable that the electromagnetic fields in even dimensions where they only exist, are intrinsically relativistic when their equations are linear, like this occurs in 3+1.

One may similarly treat polytropes ($p = A\mu^\gamma$), though in this case the Lagrangian is determined only implicitly, like this is known for $D = 3+1$, [21].

In [21] it was found that in the special relativistic approximation (the only approximate case considered in [21]), the low-amplitude mass density

perturbations of a relativistic perfect fluid (thus the relativistic sound) propagate with exactly the same velocity as it is known in the non-relativistic phenomenological theory. Using the traditional polytrope state equation, one gets then for the acoustic velocity, for example, in the air the standard expression $c_s = \sqrt{\gamma p/\mu}$. The same approach applied to the equation of state (3.6) permits to naturally introduce the concept of the stiff matter equalizing the acoustic velocity to that of light (in our units, $c = 1$). These conclusions are universal for all D's.

3.2 Perverse fluids

This is the case when $f \cdot f < 0$ (the (co)vector dual to the field tensor is spacelike), so that it can be normalized as

$$l = \frac{f}{\sqrt{-f \cdot f}}, \quad l \cdot l = -1. \tag{3.7}$$

It was just mentioned in [21] as the tachyonic (abnormal) fluid. The stress-energy tensor of this $(D-2)$-form field reads

$$T_\alpha^\beta = -L\delta_\alpha^\beta + 2I\frac{dL}{dI}\tilde{b}_\alpha^\beta, \quad \tilde{b}_\alpha^\beta = \delta_\alpha^\beta + l^\beta l_\alpha \tag{3.8}$$

where $\tilde{b}_\alpha^\beta$ is the projector on the local subspace orthogonal to the l congruence. It is clear that this subspace is timelike, and every vector in it is eigenvector of T_α^β with one and the same eigenvalue $(-\tilde{p})$, while l is an eigenvector corresponding to the non-degenerate eigenvalue $\tilde{\mu}$. These eigenvalues have the same structure as those in (3.4) which is then rewritten with tildes, but they now pertain to another combination of eigenvectors, so that a *tilde* is put over the letters denoting them. Below an example of the (3+1)-spacetime is considered.

If we orthonormalize the three linearly independent eigenvectors related to $\tilde{p}$, and admit l as the fourth unit vector, the natural tetrad is formed with respect to which the description of a perverse fluid should look most simple. There is also a symmetry (isotropy) in the local section orthogonal to l which may be of use in simplifying the field equations; this suggests, in particular, that a kind of rotation could be most probably introduced (if one looks for rotating solutions) which involves a combination of $l + \omega dt$, and not $dt + \omega d\phi$ as this is the case for rotating perfect fluids. Let us write the tetrad $\theta^{(\alpha)}$ so

that the first ($\alpha = 0$) covector, as well as the next two, will correspond to the local subspace orthogonal to l, while $\theta^{(3)} = l$. Then the stress-energy tensor will take the diagonal form

$$T_{(\alpha)(\beta)}\theta^{(\alpha)} \otimes \theta^{(\beta)} = \tilde{p}\left(\theta^{(0)} \otimes \theta^{(0)} + \theta^{(1)} \otimes \theta^{(1)} + \theta^{(2)} \otimes \theta^{(2)}\right) + \tilde{\mu} l \otimes l. \quad (3.9)$$

The issue of the equation of state of perverse fluids is very much the same as that of perfect fluids (see above and in [21]). Some differences however arise in the consideration of propagation of perturbations (special relativistic theory). It can be considered with 0th approximation for l being either dz or $rd\phi$ (in the last case, a transition to the Cartesian coordinates can be carried out globally, but, since this approach is an approximation to general relativistic theory, at any point being not at the origin, thus $r \neq 0$, one can take other tangent frame in a vicinity of that point, and pretend to use there Cartesian frame; the only exception is the "singular" point at the origin which we shall not discuss here). So let us consider $f = dz + \delta f$ and $I = -1 - 2\delta f_z$ (the second-order term is neglected here and subsequently). The pseudopotential Φ can be introduced due to the field equation (3.5) where, of course, I should be changed by $(-I)$ for a perverse fluid. Thus

$$d\Phi \equiv \frac{dL}{d(-I)} f = \left[\frac{dL}{d(-I)} dz + \frac{dL}{d(-I)} \delta f + 2\frac{d^2 L}{d(-I)^2} \delta f_z dz\right]_{I=-1},$$

the expression which yields two equations,

$$\left[\frac{dL}{d(-I)} + 2\frac{d^2 L}{d(-I)^2}\right]_{I=-1} (\delta f_z)_{,a} = \left[\frac{dL}{d(-I)}\right]_{I=-1} (\delta f_a)_{,z}$$

and

$$\left[\frac{dL}{d(-I)}\right]_{I=-1} (\delta f_a)_{,b} = \left[\frac{dL}{d(-I)}\right]_{I=-1} (\delta f_b)_{,a}$$

($a, b = 0, 1, 2$) where the indices a and b pertain to the timelike section, thus not containing z-component. The last equation is satisfied when

$$\delta f_a = \left[\frac{dL/dI + 2d^2 L/dI^2}{dL/dI}\right]_{I=-1} \left(\int \delta f_z dz + \phi\right)_{,a};$$

there are two functions, δf_z and $\phi(t, x, y)$, which are still undetermined. We use now the fact that δf (as well as f itself) is divergenceless: this means that

$$-\delta f^z_{,z} = -\delta f^a_{,a} = -\delta f_{t,t} + \delta f_{x,x} + \delta f_{y,y} =$$

$$\left[\frac{dL/dI + 2d^2L/dI^2}{dL/dI}\right]_{I=-1} \tilde{\Delta}\left(\int \delta f_z dz + \phi(t,x,y)\right)$$

where $\tilde{\Delta} = \partial^2/\partial x^2 + \partial^2/\partial y^2 - \partial^2/\partial t^2$ is an analogue of the Laplacian [or, truncated D'Alembertian] operator (in a timelike hypersurface of subspace with coordinates x^a). Taking the derivative of the both sides of the last relation with respect to z, we arrive at

$$\frac{\partial^2 \delta f_z}{\partial z^2} + \left[\frac{dL/dI + 2d^2L/dI^2}{dL/dI}\right]_{I=-1} \tilde{\Delta}\delta f_z = 0.$$

But it is worth dividing this equation by the constant coefficient before $\tilde{\Delta}$ in order to directly see with what velocity do propagate the perturbations in different directions in this obviously anisotropic world; then we get the squared velocity as a coefficient before the second partial derivative with respect of the corresponding spatial coordinate (the coefficient before $\tilde{\Delta}$ will be equal to unity). Thus we finally find that

$$\left\{\left[\frac{dL/dI}{dL/dI + 2d^2L/dI^2}\right]_{I=-1} \frac{\partial^2}{\partial z^2} + \tilde{\Delta}\right\}\delta f_z = 0. \qquad (3.10)$$

Consequently, the perturbations propagate in a perverse fluid in all directions except that which corresponds to the non-degenerate eigenvalue, with the velocity of light, while in this last (here, z) direction this velocity is inverse to the acoustic one being characteristic to a perfect fluid with the same equation of state, now — for the perverse fluid — applied to $\tilde{p}$ and $\tilde{\mu}$ (the velocities are given in the units of the velocity of light, thus they are dimensionless). There is still another distinction from the perfect fluid case: for the perverse fluid the perturbation whose propagation is considered, is *not* that of the mass density, but of the z component of the anisotropic pressure (here, $\tilde{\mu}$).

We see that the only concept which remains unchanged is that of the stiff matter (with the velocity of light, $c = 1$, for the acoustic-type waves in this fluid). Otherwise, in order to obtain subluminal velocities, one has to consider other part of the range of the parameter k which for a perfect fluid would correspond to crucially unphysical cases outside the stiff matter states. This fact is in a complete agreement with the interpretation of perverse fluids as "tachyonic" fluids. The "only" hard question in this interpretation is why a particular concrete spatial direction (here, z) is singled out, when if a stochastic motion of tachyons is being considered, there should be a spatial

isotropy in the overall picture. The theory, especially if we go to the mentioned "other part" of the range of the parameter k, is highly nonlinear, and there is no way to take a meaningful superposition of all possible directions of the axes here named as z. This is clearly a conflict between the usual concepts of physical objects and the tachyonic ones; the only case which can be meaningfully considered in this connection, is the case of linear equations, with $k = 1$, when a superposition of solutions is automatic. Probably, from this starting point one should begin the formulation of the statistical model of perverse fluids. Otherwise these theoretical objects should be taken as some formal concepts, though sufficiently well described in general relativistic field theory, and one has to look for further results following from this theory in order to come to a better understanding of this subject.

An amazing situation has been however developed in the (2+1)-spacetime theory where one of the best studied solutions is that with a perverse fluid (though generally misinterpreted as an "electric field").

3.3 Null fluids (coherent radiation)

In the null, or radiation case the vector f is null, $I = 0$, and the respective eigenvalue is equal to zero. The stress-energy tensor cannot be brought to a diagonal form, it always contains a nontrivial flux component. However this component can be made as small as one wishes (the Doppler effect), only its vanishing occurs for the degenerate (forbidden) transformation to the velocity of light in the direction of the flux.

In fact, the stress-energy tensor of the null fluid reads $T_\alpha^\beta = \lambda f_\alpha f^\beta$. Now f is not a simple eigenvector: there exist other $D-2$ eigenvectors corresponding to the eigenvalue zero, and these are spacelike vectors (say, l_a, $a = 3, \dots, D$) orthogonal to f. Let us rename f as v; thus $v \cdot v = 0$, $l_a \cdot l_b = -\delta_b^a$, $v \cdot l_a = 0$, while

$$T_\alpha^\beta = \lambda v_\alpha v^\beta, \;\; T_\alpha^\beta v^\alpha = 0 = T_\alpha^\beta l_a^\alpha. \tag{3.11}$$

We choose the last vector, w, to be null, orthogonal to l_a, and normalized with respect to v as $w \cdot v = 1$. Then the two mutually related bases (one covector, $\theta^{(\alpha)}$, and another vector one, $X_{(\alpha)}$) can be introduced as

$$\theta^{(0)} = \underline{w}, \;\; \theta^{(1)} = \underline{v}, \;\; \theta^{(a)} = \underline{l}_a \tag{3.12}$$

and

$$X_{(0)} = \overline{v}, \;\; X_{(1)} = \overline{w}, \;\; X_{(2)} = -\bar{l}_a, \tag{3.13}$$

thus $\underline{\theta}^{(\beta)} \cdot \overline{X}_{(\alpha)} = \delta^\beta_\alpha$ (the underlined objects being covectors and overlined, vectors).

In anticipation, it may be mentioned that here the rotating field can easily be described, as this is the case for the perfect and perverse fluids. Then the (co)vector w will acquire an additional term proportional to v (the coefficient being a function of coordinates), *cf.* 4.2.

4 The Machian field

The $(D-1)$-form field will be described in D by the potential

$$C = \frac{1}{(D-1)!} C_{\mu_1 \ldots \mu_{D-1}} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{D-1}} \tag{4.1}$$

yielding the skew-symmetric field tensor (D-form)

$$W = dC. \tag{4.2}$$

Its invariant reads

$$K = *(W \wedge *W) \equiv \frac{1}{D!} W_{\alpha_1 \ldots \alpha_D} W^{\alpha_1 \ldots \alpha_D} \equiv \tilde{W}^2; \tag{4.3}$$

we shall use it to construct the Lagrangian of a free $(D-1)$-form field. Here

$$\tilde{W} := *W = \frac{1}{D!} W_{\alpha_1 \ldots \alpha_D} E^{\alpha_1 \ldots \alpha_D}, \text{ thus } W_{\alpha_1 \ldots \alpha_D} =: \tilde{W} E_{\alpha_1 \ldots \alpha_D}, \tag{4.4}$$

$\tilde{W}$ being axial scalar (often called "pseudoscalar").

4.1 The cosmological term

For the $(D-1)$-form field with its invariant (4.3), the stress-energy tensor takes the form

$$T^\beta_\alpha = \left(2K \frac{dL}{dK} - L \right) \delta^\beta_\alpha, \tag{4.5}$$

exactly coinciding with this tensor for the 3-form field in 3+1 (but now with D-dimensional indices). A similar situation repeats for the field equations: they reduce to

$$K^{1/2} \frac{dL}{dK} = \text{const.} \tag{4.6}$$

The further conclusions are the same as in [21] for the 3-form field, and we repeat them here in short. If $L \sim K^{1/2}$, all $(D-1)$-forms C identically satisfy (4.6), while the stress-energy tensor identically vanishes. Otherwise, K should be constant, and the stress-energy tensor (4.5) becomes proportional to δ_α^β with a constant coefficient obviously identifiable with the cosmological constant Λ. The case $L \sim K^{1/2}$ corresponds to $\Lambda = 0$, this being the intrinsically relativistic $(D-1)$-form field in D, offering an alternative interpretation to the cosmological constant problem (see [21]). Since the 2-form C also permits to introduce rotating fluids in 2+1 (similarly to this role of 3-form field in 3+1, though we leave here this subject without further consideration), we are inclined to relate this field to the fundamental cosmological Machian field (probably, of the type of that proposed by Sakurai, [25]).

4.2 Rotating systems

Turning to the rotating fluid case, one has to generalize the dynamical equation, for example that previously taken in the form (3.5), so that it will describe a rotating congruence f or, equivalently, u. The (3+1)-dimensional case was already discussed in [21, 22]. Below we shall consider only the case of the (2+1)-spacetime where we have to add to the Lagrangian $L(I)$ a new term, say, $M(J)$, where a new invariant involving $\tilde{W}$ [see (4.4)] as well as $A_\mu f^\mu$, reads

$$J := W_{\lambda\mu\nu} A^\lambda F^{\mu\nu} \equiv \tilde{W} E_{\lambda\mu\nu} A^\lambda F^{\mu\nu} \equiv 2\tilde{W} A^\lambda f_\lambda \tag{4.7}$$

[a product of two axial scalars (pseudoscalars)]. Due to the complete antisymmetrization of the product $A \wedge F$ in (4.7) involving three indices, the additional term appearing in the stress-energy tensor, will be of the type of (4.5):

$$T_\alpha^\beta = \left(2J \frac{dM}{dJ} - M \right) \delta_\alpha^\beta, \tag{4.8}$$

thus it will vanish if $M \sim J^{1/2}$, or if $J \frac{dM}{dJ}$ vanishes simultaneously with M due to some property of M, as this will be the case for the problems considered in Section 6. The new 1-form field equation is

$$d\left(\frac{dL}{dI} f + \frac{dM}{dJ} \tilde{W} A \right) = -\frac{dM}{dJ} \tilde{W} F, \tag{4.9}$$

thus $d\left(\frac{dM}{dJ} \tilde{W} F \right) = 0$ is a condition on $\tilde{W}$.

For the 2-form field, the additional term in the left-hand side of the equation (4.6), emerges:

$$2\frac{dM}{dJ}A^{\lambda}f_{\lambda} \tag{4.10}$$

[its sum with the left-hand side of (4.6) to be a constant; however, this contribution vanishes at least under a sufficiently general class of natural assumptions, so we do not write here the complete sum of terms from (4.6) and (4.10)]. Thus from (4.9) and (4.6) plus (4.10) one can calculate the function $\tilde{W}$ which was in fact still arbitrary, and obtain the rotation characteristics of the fluid. This completes the introduction of rotation in the theory of perfect fluids in 2+1 (*cf.* a different method used in 3+1, [21]; the important differences are here due to the change of dimensionality).

5 Electromagnetism from the systematic viewpoint

As in the case of a perfect fluid, we postulate here in D dimensions essentially the same properties of the stress-energy tensor as they are in 3+1 for the field under consideration (now, Maxwell's field). The only differences are those which are due to the other dimensionality. It is easy to see that the same distribution of eigenvalues and eigenvectors (the degenerate pairs of eigenvalues, and D eigenvectors equally divided between these eigenvalues) is possible only in even number of dimensions, thus $D = 2m$. This means that the intensity tensor F of the field should be an m-form [hence the potential tensor A is an $(m-1)$-form], and in accordance with the terminology of the theory we call this field the $(m-1)$-form field. In general, this field gives two invariants, $I_1 = F \cdot F$ and $I_2 = F \cdot *F$. $*$ means here the Hodge star (dual conjugation) which merely rearranges the components of F when passing to $*F$, as this is the case in 3+1. Similarly, the second invariant in general reduces to a covariant divergence, thus not being of use in obtaining the linear Euler–Lagrange equations (it can result then merely in a surface term). The analogy with the usual Maxwell theory further spreads to the dynamical and structure field equations. The stress-energy tensor is equally distributed between two m-dimensional subspaces, and in the linear theory ($k = 1$) it manifests intrinsically relativistic features — always when this Maxwell-type field can exist (for all $D = 2m$, see Table 1). It is obvious that these

properties cannot occur by a coincidence, so this should express a profound law of nature. One has all reasons to admit that this is a fundamental field which singles out the even-dimensional spacetimes in the physical picture of universe.

In 1+1, and only then, the Maxwell-type field is simultaneously a fluid field (its magnetic type coinciding with a perfect, and electric, with a perverse fluid, the both being intrinsically relativistic), and, moreover, this is a scalar field. In 2+1 and 4+1 (as in all other odd-dimensional spacetimes) the Maxwell-type field simply does not exist (see the Theorem in the Appendix B).

6 Some exact solutions

In our approach some new ways can be used to find exact Einstein–Maxwell and Einstein–Euler solutions. In particular, when fluids are described via $(D-2)$-form fields, the equations of state corresponding to the specific invariant-dependence of Lagrangians $[L(I)]$ provide additional algebraical equations which give new relations between unknown functions. Another way leading from already known solutions, for example, in 3+1, to lower-dimensional (say, 2+1) solutions, is the use of spacetime sections of the former spacetimes, sometimes with a redetermination of functions (see [23] as well as the null solutions given below). Still another method of finding new exact solutions, this time in higher-dimensional spacetimes, is a hybridization (sometimes, inbreeding- and chimaera-engineering) of already-known lower-dimensional solutions or their sections, in general involving a redetermination of functions.

Below some examples are given of new solutions in 2+1, the incoherent dust and coherent null fluid solutions.

6.1 Generalities

First we consider the general Einstein-Euler equations with a natural metric ansatz. The case when $I > 0$, as we already know, describes perfect fluids. In the curvature coordinates (Synge's terminology) the appropriate orthonormal tetrad basis is

$$\theta^{(0)} = e^{\alpha}(dt - \Phi d\phi),\ \theta^{(1)} = e^{\beta} dr,\ \theta^{(2)} = r d\phi, \tag{6.1}$$

α, β, and Φ being functions of r. Then

$$ds^2 = e^{2\alpha}(dt - \Phi d\phi)^2 - e^{2\beta}dr^2 - r^2 d\phi^2, \quad \sqrt{g} = re^{\alpha+\beta}. \tag{6.2}$$

It is easy to calculate (using, for example, the Cartan exterior forms formalism) the Ricci tensor components,

$$\begin{aligned}
R_{(0)(0)} &= -\frac{1}{r}\left(r\alpha' e^{\alpha-\beta}\right)' e^{-(\alpha+\beta)} - \frac{1}{2r^2}\Phi'^2 e^{2(\alpha-\beta)}, \\
R_{(1)(1)} &= \left(\alpha' e^{\alpha-\beta}\right)' e^{-(\alpha+\beta)} - \frac{\beta'}{r}e^{-2\beta} - \frac{1}{2r^2}\Phi'^2 e^{2(\alpha-\beta)}, \\
R_{(2)(2)} &= \frac{1}{r}\left(e^{\alpha-\beta}\right)' e^{-(\alpha+\beta)} - \frac{1}{2r^2}\Phi'^2 e^{2(\alpha-\beta)}, \\
R_{(0)(2)} &= \frac{1}{2}\left(\frac{1}{r}\Phi' e^{(3\alpha-\beta)}\right)' e^{-(2\alpha+\beta)},
\end{aligned}$$

as well as the scalar curvature $R = R_{(0)(0)} - R_{(1)(1)} - R_{(2)(2)}$. Of course, the curvature coordinates are not applicable when Nariai-type spacetimes are considered. The convenience of the proper (eigenvector) basis is that with respect to it the stress-energy tensor takes the diagonal form, hence $R_{(0)(2)} = 0$ and $R_{(1)(1)} = R_{(2)(2)}$. The both of these equations are easily integrated yielding

$$\frac{1}{r}\Phi' e^{3\alpha-\beta} = \omega = \text{const} \tag{6.3}$$

and

$$\frac{1}{r}\left(e^{\alpha}\right)' e^{-\beta} = C = \text{const}, \tag{6.4}$$

respectively.

The remaining Einstein equations read

$$G_{(0)(0)} \equiv \frac{1}{2}\left(R_{(0)(0)} + R_{(1)(1)} + R_{(2)(2)}\right) = -\varkappa\mu \tag{6.5}$$

and

$$\frac{1}{2}\left(G_{(1)(1)} + G_{(2)(2)}\right) \equiv \frac{1}{2}R_{(0)(0)} = -\varkappa p. \tag{6.6}$$

In the description of a perfect fluid they are usually treated as "definitions" of the mass density and pressure (though when a specific equation of state is assumed, their combination gives an additional restriction on the metric coefficients).

Let us consider for a while the flat 2+1 spacetime in polar coordinates,

$$ds_0^2 = dt^2 - dr^2 - r^2 d\phi^2, \tag{6.7}$$

with

$$A = qa(r)d\phi \tag{6.8}$$

where q is a constant, and see which function $a(r)$ would fulfil the 1-form field equation in a non-self-consistent problem (without taking into account Einstein's equations). First of all we observe that

$$F = dA = qa' dr \wedge d\phi, \tag{6.9}$$

so that

$$I = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} = \frac{q^2 a'^2}{r^2} > 0 \tag{6.10}$$

(clearly, the perfect fluid case). Let also $L = \sigma I^k$, thus

$$\sqrt{g} \frac{dL}{dI} F^{\mu\nu} = \sigma k \left(\frac{qa'}{r}\right)^{2k-1} \left(\delta_r^\mu \delta_\phi^\nu - \delta_\phi^\mu \delta_r^\nu\right). \tag{6.11}$$

Then the 1-form field equation reads

$$\left(\sqrt{g} \frac{dL}{dI} F^{\mu\nu}\right)_{,\nu} = -\sigma k \left[\left(\frac{qa'}{r}\right)^{2k-1}\right]' \delta_\phi^\mu = 0 \tag{6.12}$$

(no rotation is involved). The solution is $a' = r$ (the integration constant is incorporated into q); another possible solution is $k = 1/2$, but this is the case of an incoherent dust which will be later discussed separately, so we shall now consider the first alternative.

Now, returning to our more general problem, we plausibly postulate the 1-form field potential and the corresponding field tensor to be

$$A = \frac{1}{2} q r^2 d\phi, \quad F_{\mu\nu} = qr \left(\delta_\mu^r \delta_\nu^\phi - \delta_\mu^\phi \delta_\nu^r\right) \tag{6.13}$$

(the second integration constant in $a(r)$ — see (6.12) — corresponds to addition of an exact form to A and is dropped). This ansatz is similar to the Horský–Mitskievich method of constructing exact charged solutions in 3+1 (see [11] and [28]), though the Killing vector ξ approach is not automatically

applicable directly to the 1-form field potential A due to the (in general) nonlinear nature of the field.

In (6.13) we find components of the covariant field tensor in the spacetime metricized by (6.2); its independent nontrivial contravariant components are

$$F^{12} = \frac{q}{r} e^{-2\beta}, \quad F^{10} = \frac{q}{r} \Phi e^{-2\beta}. \tag{6.14}$$

Since $f = *F$,

$$f_\mu = q e^{\alpha-\beta} \left(\delta^t_\mu - \Phi \delta^\phi_\mu \right), \quad f^\mu = q e^{-(\alpha+\beta)} \delta^\mu_0. \tag{6.15}$$

This means, in particular, that

$$A_\mu f^\mu = 0 \tag{6.16}$$

[*cf.* the remarks on vanishing of M in the Subsection 4.2, as well as the expression (4.10)]. The relativistic velocity of the fluid is the normalized vector f,

$$u = \theta^{(0)} = e^\alpha (dt - \Phi d\phi), \quad u^\mu = e^{-\alpha} \delta^\mu_0, \tag{6.17}$$

and the field invariant is

$$I = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} = f_\mu f^\mu = f_0 f^0 = q^2 e^{-2\beta}. \tag{6.18}$$

This expression as well as those to be deduced below will prove to be of importance in finding exact solutions of the complete system of field equations. The usual treatment of perfect fluids concentrates essentially on Einstein's equations, but we shall see that inclusion of 1- and 2-form field equations and concrete expressions of Lagrangian and stress-energy tensor components, makes calculations much easier.

6.2 Rotating solutions

As to the Lagrangians and field equations, we admit the relations given in Subsection 4.2 with vanishing invariant J whose derivatives with respect to A_μ and $F_{\mu\nu}$ are however different from zero. Thus the 2-form field equation [(4.6) plus (4.10)] is satisfied trivially (we shall not use the free field Lagrangian in this case) and the 1-form field equation takes the form

$$d\left[\frac{dL}{dI} 2e^{\alpha-\beta}(dt - \Phi d\phi) + \frac{dM}{dJ} \tilde{W} r^2 d\phi\right] = -2\frac{dM}{dJ} \tilde{W} r \, dr \wedge d\phi. \tag{6.19}$$

This equation implies (for general $a(r)$ as well)

$$\frac{dL}{dI}e^{\alpha-\beta} = -N = \text{ const.}, \tag{6.20}$$

a new integral when $\Phi \neq 0$, and

$$\left(\frac{dM}{dJ}\tilde{W}r^2 + 2N\Phi\right)' = -2\frac{dM}{dJ}\tilde{W}r. \tag{6.21}$$

It is clear that rotation will not disappear only if $M = \lambda J$ with $\lambda = $ const; we put $\lambda = 1$ without infringing generality of our considerations. Thus

$$\tilde{W} = -\frac{2N}{r^4}\int \Phi' r^2 dr, \tag{6.22}$$

while the function Φ, as well as another unknown function, have to be found from Einstein's equations. In 3+1 we call the conditions corresponding to (6.20) and (6.21) in the Horský–Mitskievich approach, the Maxwell conditions (they follow there from Maxwell's equations); let us baptize them in 2+1 as the Euler conditions.

Turning to the simplest equation of state (3.6) when $L = -\sigma I^k$, thus $dL/dI = kL/I$, and taking into account (6.18) and (6.20), we find that

$$\mu = -L = \frac{Nq^2}{k}e^{-\alpha-\beta}. \tag{6.23}$$

Thus

$$p = \frac{2k-1}{k}q^2 N e^{-\alpha-\beta}. \tag{6.24}$$

However there is another way to express L: directly from (6.18),

$$L = -\sigma I^k = -\sigma q^{2k} e^{-2k\beta}. \tag{6.25}$$

Comparing the two expressions of L, (6.23) and (6.25), we come to a remarkable algebraic relation between α and β,

$$\exp(\alpha - (2k-1)\beta) = \frac{2N}{2k\sigma}q^{2(1-k)}. \tag{6.26}$$

This means that we need to determine only two functions to solve our problem, say, α and Φ (in the non-rotating case there remains only one function to be found).

6.3 Incoherent dust

Incoherent dust ($k = 1/2$, thus $p = 0$) is here a special case for which it follows from (6.26) that $\alpha = \text{const}$, but this, according to (6.6), excludes the possibility of rotation for the admitted form of the metric, with the both ω and C being equal to zero. Thus in a 2+1 spacetime there exists a continuous family of *static* non-rotating dust solutions (among them a continuous set of singularities-free ones), in fact with an arbitrary spherically symmetric distribution of mass density, in an acute contrast to the situation familiar in the 3+1 spacetime. In this case, the function $\beta(r)$ is simply not present in eq. (6.26). The only surviving Einstein equation is (6.5) which now takes the form

$$\left(e^{-2\beta}\right)' = -2\varkappa\mu(r)r. \tag{6.27}$$

This possibility of existence of static dust solutions in 2+1 can be related to the coefficient $D-3$ in the weak-field approximation for $R^{(0)}_{(0)}$ and in the equation for the "Newtonian potential" in this case, see (A.3) and (A.9) respectively: the Newtonian potential simply vanishes in 2+1 for any dust distribution.

It is clear that the equations obtained above, are much more general that we needed in the description of dust solutions. Indeed, they well serve in finding many other solutions (*e.g.*, the 2+1 analogue of the Gödel rotating world); these results will be presented elsewhere.

6.4 Null solutions

Solutions for null sources correspond in 3+1 to those which describe spacetimes filled with coherent radiation (null fluid; this latter name is also applicable to *stationary* null fields, in particular such types of electromagnetic and fluid fields). In 3+1 this includes pp-wave solutions, as well as the Robinson–Trautman null radiation and Einstein–Maxwell fields, known only in the absence of the cosmological term [13]. Here we find that in 2+1 this limitation is lifted.

It is easy to see that the gravitational field with

$$ds^2 = A(u)y^2du^2 + 2dudv - dy^2 \tag{6.28}$$

corresponds to

$$R_{0202)} = R_{00} = R_{00} - \frac{1}{2}g_{00}R = A(u),\ R = 0, \tag{6.29}$$

all other independent components of these tensors being equal to zero. This is exactly the 2+1 Einstein–Euler pp-wave solution, its 3+1 counterpart being a direct product of this 2+1 spacetime and a one-dimensional space. It is worth mentioning that the spacetime (6.28) is conformally flat in the sense of the Cotton–Schouten–York tensor, while its 3+1 counterpart is, of course, of type N.

A hybrid of (6.28) and the Robinson–Trautman solution in the tetrad form

$$\theta^{(0)} = e^{2\sqrt{\Lambda}y}du - \frac{A(u)y^2}{2}dv,\ \theta^{(1)} = dv,\ \theta^{(2)} = dy, \tag{6.30}$$

$$g_{(\alpha)(\beta)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{6.31}$$

yields the non-zero independent components of the Riemann–Christoffel tensor

$$R_{(0)(1)(0)(1)} = R_{(0)(2)(1)(2)} = -\Lambda,\ R_{(0)(2)(0)(2)} = A(u)(2\sqrt{\Lambda}y+1)e^{-2\sqrt{\Lambda}y}, \tag{6.32}$$

those of the Ricci curvature,

$$R_{(0)(0)} = A(u)(2\sqrt{\Lambda}y + 1)e^{-2\sqrt{\Lambda}y},\ R_{(0)(1)} = -2\Lambda,\ R_{(2)(2)} = 2\Lambda, \tag{6.33}$$

the scalar curvature being $R = -6\Lambda$, and non-zero components of the Einstein conservative tensor being

$$G_{(0)(0)} = A(u)(2\sqrt{\Lambda}y + 1)e^{-2\sqrt{\Lambda}y},\ G_{(0)(1)} = \Lambda,\ G_{(2)(2)} = -\Lambda. \tag{6.34}$$

This is the same type of wave as (6.28), but with the cosmological constant Λ. It is obvious that when $\Lambda \to 0$ the metric defined by (6.30) and all its concomitants reduce to those of (6.28).

A modification of the Robinson–Trautman Einstein–Maxwell (radiation) solution to 2+1 in the coordinates u, r, ϕ (the first two of them pertaining to the null part of the basis) is

$$\theta^{(0)} = \left(-F(u) + \frac{1}{2}\Lambda r^2\right) du + dr,\ \theta^{(1)} = du,\ \theta^{(2)} = r d\phi. \tag{6.35}$$

Its non-zero independent concomitants are: the Riemann–Christoffel tensor,

$$R_{(0)(1)(0)(1)} = R_{(0)(2)(1)(2)} = -\Lambda,\ R_{(1)(2)(1)(2)} = \frac{F'(u)}{r}, \tag{6.36}$$

the Ricci tensor,

$$R_{(0)(1)} = -R_{(2)(2)} = -2\Lambda,\ R_{(1)(1)} = \frac{F'(u)}{r}, \tag{6.37}$$

$R = -6\Lambda$, and Einstein's conservative tensor

$$G_{(0)(1)} = -G_{(2)(2)} = \Lambda,\ G_{(1)(1)} = \frac{F'(u)}{r}. \tag{6.38}$$

This now is a cosmological solution with null matter.

Moreover, the 3+1 Brdička solution (see [13], p. 236) which describes a conformally flat spacetime with a constant null Maxwell field (crossed electric and magnetic fields with equal intensities), has a following 2+1 counterpart:

$$\theta^{(0)} = du,\ \theta^{(1)} = dv,\ \theta^{(2)} = A(u)dy \tag{6.39}$$

with the same tetrad metric (6.31). In this case

$$R_{(0)(2)(0)(2)} = R_{(0)(0)} = G_{(0)(0)} = \frac{A''(u)}{A(u)}, \quad R = 0 \tag{6.40}$$

(all other independent components vanish), thus for $A(u) \sim \cosh(\omega u)$ with a constant ω, the same meaning as in 3+1 (only with the reinterpretation of 2+1 sources as fluids, perhaps a mixture of them) persists, but for a general $A(u)$ this metric describes a wider class of null solutions, including wavelike ones (however without the Λ term).

7 New prospects of the systematic field-theoretic approach (conclusions)

It is now possible to outline a scenario of the most elementary stage of the universe evolution from the very first step when there are only two spacetime dimensions ($D = 1+1$), the model which I had proposed many years ago [17] but which has taken a definitive shape only within the theory formulated in this talk.

In 1+1, there is no real distinction between space and time, so that the only invariant frame is that of the light cone, the future and the past being purely conditional. It is most plausible that, similarly, only intrinsically

relativistic and null objects (fields) can exist there. These are two fields: 0-form and 1-form ones. From Table 1 we see that they are the linear Maxwell-type and ghost Machian fields, respectively. The intrinsically relativistic 1-form Machian field in 1+1 is arbitrary since it is automatically a ghost field (the stress-energy tensor identically vanishing, $\Lambda = 0$). Its intensity 2-form is proportional to the 2-dimensional Levi-Civitą symbol giving the simplectic (skew) metric tensor, like that which is used in the spinor (complex) 2-space. It is still not clear if the proportionality coefficient (an arbitrary function) pertains to this metric or represents an independent pseudoscalar object (the first possibility seems to be more natural). These steps were realized without any other introduction of a metric, and now we get one directly from the theory, nothing less than from the Machian field. In the Maxwell-type field we have to use exactly this metric. In 1+1, this is a linear scalar field about which I have still nothing more to tell. But to the metric *alias* Machian field, while it is still completely unconstrained, an additional action principle could be applied, with a Lagrangian built of the very metric and its derivatives.

We see that some, probably, very simple theory works in 1+1. There should exist some mechanism responsible for glueing together pairs of elementary cells (areas) of 1+1 resulting in elements of, most probably, the twistor space. This glueing is not purely geometric, but more a topological issue which results in a four-dimensional geometry, obviously consistent with our 3+1 spacetime.

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Appendices

A The weak field limit of Einstein's equations with relativistic sources

The Newtonian approximation for the gravitational field equation does not necessarily involve admission of non-relativistic properties of the source terms in Einstein's equations: it is sufficient to merely consider the weak-field condition for gravitational field. When a source has electromagnetic nature, one simply *cannot* ignore its intrinsically relativistic properties, since there cannot be invented any non-relativistic approximation which would describe electromagnetic stress-energy-momentum complex adequately.

Starting with Einstein's equations,

$$R^{(\alpha)}_{(\beta)} - \tfrac{1}{2} R \delta^{\alpha}_{\beta} = -\varkappa T^{(\alpha)}_{(\beta)}, \tag{A.1}$$

and taking into account that $R = \frac{2\varkappa}{D-2} T$, we rewrite them in D dimensions as

$$R^{(\alpha)}_{(\beta)} = -\varkappa \left(T^{(\alpha)}_{(\beta)} - \tfrac{1}{D-2} T \delta^{\alpha}_{\beta} \right). \tag{A.2}$$

Thus for 00-component we have

$$R^{(0)}_{(0)} = -\tfrac{\varkappa}{D-2} \left[(D-3) T^{(0)}_{(0)} - T^{(i)}_{(i)} \right]. \tag{A.3}$$

For $D = 4$ the temporal part of the stress-energy tensor enters this equation symmetrically with its spatial trace. Then

$$R^{(0)}_{(0)} = -\tfrac{\varkappa}{2} \left[T^{(0)}_{(0)} - T^{(i)}_{(i)} \right], \tag{A.4}$$

and for intrinsically relativistic sources [$T^{(\alpha)}_{(\alpha)} = 0$, the $(D-1)$-spatial part $T^{(i)}_{(i)}$ has the same magnitude as the temporal term $T^{(0)}_{(0)}$, but it comes with the opposite sign]

$$R^{(0)}_{(0)} = -\varkappa T_{\mathrm{intr.rel}}{}^{(0)}_{(0)}. \tag{A.5}$$

Taking into account that $\varkappa = 8\pi G$, where G is the Newtonian gravitational constant and the velocity of light $c = 1$, we find that in the intrinsically relativistic case $R^{(0)}_{(0)}$ is twice greater than in the non-relativistic case (when $T^{(i)}_{(i)} \approx 0$).

For $D = 3$ the temporal part is simply absent. But when $T^{(\alpha)}_{(\alpha)} = 0$, we have $T^{(i)}_{(i)} = -T^{(0)}_{(0)}$ (the intrinsically relativistic case). Thus there appears the same coefficient $\varkappa = 8\pi\gamma$, as it was the case for $D = 4$. This means that in 2+1 the Newtonian-type potential can appear only in a distribution of intrinsically relativistic matter [see (A.3)]. Moreover, one has to keep in mind the fact that there is no interaction between islets of matter submerged in vacuum, so that in order to come to an analogue of the Newtonian potential one has to consider as the zeroth approximation, an anti-de Sitter substratum. However, an overall distribution of matter is also admissible, especially when it asymptotically tends to zero (to a vacuum, thus to the flat 2+1 spacetime), and this seems to be more acceptable than the cosmological term which inevitably does not lead to such an asymptotic behaviour.

Let us consider here a static space-time with $g_{00} = 1 + 2\Phi_{\mathrm{N}}$, $\Phi_{\mathrm{N}} \ll 1$. The Newtonian approximation is now found from the geodesic equation for a non-relativistic test particle. One has to express $R^{(0)}_{(0)}$ in terms of the Newtonian potential Φ_{N} (in fact, its derivatives) neglecting the higher order terms (non-linear in the Newtonian potential and other corrections to the flat — here, Cartesian — part of the metric coefficients in a coordinated basis, all these corrections including Φ_{N} being considered as infinitesimals of the first order of magnitude).

We choose now a static 1-form basis in spacetime,

$$\theta^{(0)} = \mathrm{e}^{\alpha} dt, \quad \theta^{(k)} = g^{(k)}{}_{j} dx^{j}, \tag{A.6}$$

this choice being here general enough. Then, taking the inverse triad as $g_{(k)}{}^{j}$, so that $dx^{j} = g_{(k)}{}^{j}\theta^{(k)}$, $dt = \mathrm{e}^{-\alpha}\theta^{(0)}$, we find $d\theta^{(0)} = \alpha_{,j} g_{(k)}{}^{j}\theta^{(k)} \wedge \theta^{(0)}$, from where it is easy to calculate the necessary components of 1-form connections (in this static case): $\omega^{(0)}{}_{(l)} \equiv \omega^{(l)}{}_{(0)} = \alpha_{,j} g_{(l)}{}^{j}\theta^{(0)}$. From the second Cartan structure equations,

$$\Omega^{(\alpha)}{}_{(\beta)} = d\omega^{(\alpha)}{}_{(\beta)} + \omega^{(\alpha)}{}_{(\gamma)} \wedge \omega^{(\gamma)}{}_{(\beta)},$$

neglecting non-linear terms (since in this Appendix the weak-field approximation is considered only), we find that

$$\Omega^{(0)}{}_{(l)} \approx \left(\alpha_{,j} g_{(l)}{}^{j}\right)_{,k} g_{(i)}{}^{k}\theta^{(i)} \wedge \theta^{(0)} + \alpha_{,j} g_{(l)}{}^{j} \alpha_{,i} g_{(k)}{}^{i}\theta^{(k)} \wedge \theta^{(0)} \tag{A.7}$$

(the last term is written here for symmetry reasons, though it should be, of course, omitted). Now,

$$R^{(0)}_{(0)} = g^{(l)(k)} R^{(0)}{}_{(l)(k)(0)} \approx \mathrm{e}^{-\alpha} \left(\mathrm{e}^{\alpha}\right)_{,i,j} g^{ij}$$

where $g^{ij} = -\delta^i_j$ + higher-order terms (to be neglected). Since $\mathrm{e}^{\alpha} \approx 1 + \Phi_{\mathrm{N}}$,

$$R^{(0)}_{(0)} \approx -\Delta\Phi_{\mathrm{N}} \approx -\tfrac{\varkappa}{D-2}\left((D-3)T^{(0)}_{(0)} - T^{(i)}_{(i)}\right), \quad \varkappa = 8\pi G; \tag{A.8}$$

thus, in the linear static approximation,

$$\Delta\Phi_{\mathrm{N}} \approx \frac{8\pi G}{D-2}\left((D-3)T^{(0)}_{(0)} - T^{(i)}_{(i)}\right), \tag{A.9}$$

the result coinciding in $D = 4$ with (B.1) (the fields for which $-T^{(i)}_{(i)} \ll T^{(0)}_{(0)}$, being non-relativistic, and for which $-T^{(i)}_{(i)}/(D-1) \approx T^{(0)}_{(0)}$, intrinsically relativistic). We see that for arbitrary D in the rest frame of the fluid the equation (A.9) takes the form

$$\Delta\Phi_{\mathrm{N}} \approx \frac{8\pi G}{D-2}\left[(D-3)\mu + (D-1)p\right]. \tag{A.10}$$

When a perfect fluid is considered, its energy-momentum tensor being (1.1), in the rest frame of the fluid one has to compare $(D-1)p$ and $(D-3)\mu$. The Newton–Poisson equation (A.9) takes for 3+1 the form

$$\Delta\Phi_{\mathrm{N}} \approx 4\pi G(\mu + 3p). \tag{A.11}$$

If $p \ll \mu$, the old traditional equation follows, but if the fluid represents an incoherent radiation ($p = \mu/3$), the source doubles, and in the case of a stiff matter ($p = \mu$), it quadruples. This last case is, perhaps, not quite a physical one, as, probably, all cases with $p > \mu/3$ which one may call "hyperrelativistic" ones. But the stiff matter case may attract some attention since this is a simple model which sometimes permits analytical consideration.

In 2+1 the situation changes drastically [see the discussion of the expression (A.3) above]: the case when a usual Newtonian potential exists, is shifted to intrinsically relativistic sources. One has also to keep in mind that then the behaviour of Φ_{N} should correspond to a solution of the Poisson equation with the two-dimensional Laplacian. This is, of course, not applicable to the "stiff matter" in 2+1 (when $p = \mu$), since the only exact solution of Einstein's equations existing in this case, is that with a cosmological term which is constant, thus excluding the flat spacetime asymptotics.

The weak field approximation, of course, does not affect exact results in general relativity, in particular, in cosmology. However, some traditional

principles of physics are sensitive to the approximate forms of equations, and one of the most important examples is the principle of equivalence. The conclusions drawn in this Appendix suggest a relativistic generalization of this principle, especially since the Newtonian-type potential is generated by a wide class of distribution of matter, including intrinsically relativistic and hyperrelativistic matter: the only restriction in this case consists of the weakness of the field and not the "state of motion" of the sources in Einstein's equations (especially such an intrinsic property as to be relativistic which is so often realized by static configurations when the very idea of motion is out of the question). As to the applications of this generalized principle of equivalence, it is worth pointing out the (post-) post-Newtonian approximations. Since some conclusions about validity of the principle of equivalence come from observations of stellar systems, a mere presence in them of intrinsically relativistic objects (say, high density of any kind of radiation, strong or widely distributed magnetic fields, existence of stiff matter in cores of exotic stars) would radically change interpretation of the observational data if their proper understanding depends on adequate application of approximated description.

B Intrinsically relativistic fields

We introduce here in the general case of $D = n + 1$ spacetime the concept of intrinsically relativistic objects which remain relativistic even when being "at rest" (static or stationary, when fields and not particles are considered). This concept was already used implicitly in the four-dimensional case, especially when a perfect fluid with the equation of state $p = \mu/3$ (incoherent radiation) was considered. One of the reasons to take seriously the concept of intrinsically relativistic fields (and objects) consists of appearance of factor 2 in effects of their interaction with weak gravitational fields in 3+1 dimensions. This factor was first noticed in a comparative study of the effect of bending of light rays in the gravitational field (of sun): Soldner [26] and Einstein [7] versus Einstein [8] (see [29, 27, 14, 32]). There exists, of course, also the "inverse" (in the spirit of the Newtonian third law) effect (generation of gravitational field by electromagnetic field) involving doubled electromagnetic energy density ([18, 31, 19, 20]) in the four-dimensional weak gravitational field approximation (see Appendix A):

$$\Delta\Phi_{\mathrm{N}} = 4\pi G\mu_{\mathrm{non-relat}} + 8\pi G\mu_{\mathrm{em}} \tag{B.1}$$

[this (at least) doubling occurs, of course, for any intrinsically relativistic sources, not only Maxwell's field in 3+1]. Thus the intrinsically relativistic objects are exceptional only in the sense that their relativistic nature is absolute and does not depend on the choice of reference frame in which they are observed; a concrete consideration of these properties for arbitrary dimensionality of spacetime see below.

If a single point-like object is intrinsically relativistic, it has to move with the speed of light (a null world line, $\mathbf{p}^2 = E^2$), since only this velocity is absolute (both in the special and general relativity). For a distributed matter (in particular, a field), this corresponds to vanishing of the trace of its stress-energy tensor: in certain sense, temporal and spatial parts of its energy-momentum tensor contribute equally, but with opposite signs.

The stress-energy tensor of an r-form field in general takes the form (2.5), since the Lagrangian density, as well as the function L depend on $g^{\mu\nu}$ only algebraically (the r-form potentials are considered to be independent of the metric tensor). The trace of this stress-energy tensor is (2.6). Then the intrinsically relativistic property condition $T^\alpha_\alpha = 0$ yields $L \sim I^k$, $k = \frac{D}{2(r+1)}$. Another way to deduce this expression for k, if the homogeneity law $L \sim I^k$ is already accepted, consists of equally distributing the metric tensor factors (including those which are found in $\sqrt{|g|}$) in the definition of $\mathfrak{L} = \sqrt{|g|}L$ between the field tensor components of the r-form field, with the subsequent application of the Noether theorem [15, 16]:

$$\mathfrak{T}^\beta_\alpha := \frac{\delta\mathfrak{L}}{\delta g_{\mu\nu}} g_{\mu\nu}|^\beta_\alpha \equiv \frac{\delta\mathfrak{L}}{\delta g^{\mu\nu}} g^{\mu\nu}|^\beta_\alpha = \frac{\delta\mathfrak{L}}{\delta\left(|g|^{\frac{1}{2(r+1)}} g^{\mu\nu}\right)} \left(|g|^{\frac{1}{2(r+1)}} g^{\mu\nu}\right)\Big|^\beta_\alpha . \qquad \text{(B.2)}$$

Then it is clear that the intrinsically relativistic property of the field is equivalent to vanishing of trace of the Trautman coefficient [30] $\left(|g|^{\frac{1}{2(r+1)}} g^{\mu\nu}\right)\Big|^\beta_\alpha$:

$$\left(|g|^{\frac{1}{2(r+1)}} g^{\mu\nu}\right)\Big|^\alpha_\alpha = |g|^{\frac{1}{2(r+1)}} g^{\mu\nu} \left(2 - \frac{D}{k(r+1)}\right) = \text{ (the ansatz) } = 0, \quad \text{(B.3)}$$

quod erat demonstrandum. When $k = 1$, only space-times of even number of dimensions D can fit this condition: $D = 2(r+1)$. The same condition determines the conformal invariance property of the fields.

Thus in the intrinsically relativistic case it is necessary and sufficient to

use the simplest nonlinear Lagrangian densities (see the Table 1),

$$\mathfrak{L} = \sqrt{|g|}\sigma I^k, \quad k = \frac{D}{2(r+1)}. \tag{B.4}$$

$r\backslash D$	2	3	4	5	6	7	8
0	1	3/2	2	5/2	3	7/2	4
1	1/2	3/4	1	5/4	3/2	7/4	2
2		1/2	2/3	5/6	1	7/6	4/3
3			1/2	5/8	3/4	7/8	1
4				1/2	3/5	7/10	4/5
5					1/2	7/12	2/3
6						1/2	4/7
7							1/2

Table 1. Values of k versus r and D, describing general intrinsically relativistic fields (B.4).

This table simply gives values of k; since $0 \leqslant r \leqslant D-1$, the lower left corner consists of blank spaces only: the "missing" r-form fields are either trivially exact ones, or equal to zero.

Now one may consider intrinsically relativistic fields of any rank r for every dimension D, this being possible at the cost of admission of non-linear fields ($k \neq 1$). When $k = 1$, a linear field is realized (*cf.* the Table 1). It is easy to see that all these intrinsically relativistic fields (for all corresponding values of D) automatically possess the property of conformal invariance. When $T^\alpha_\alpha < 0$ (fields which are "more relativistic" than, for example, the 3+1 Maxwell field and the incoherent radiation are), we can speak on intrinsically hyperrelativistic fields, corresponding in their ultimate case to the stiff matter.

The concept of a fundamental field in D dimensions analogous to the 3+1 Maxwell field, can be now formulated as that of a linear intrinsically relativistic field. Thus in odd-dimensional spacetimes there is no room for Maxwell-like fields (for example, in 2+1 there is no analogue of the electromagnetic field in its proper sense), and in the even-dimensional ones, such fields generally should not be described by a 1-form potential (which is the case in 3+1 only). In 5+1-dimensional spacetime, this will be a 2-form field; in 7+1, 3-form field; in 9+1, 4-form field, and so on. They all admit the dual

conjugation of the respective field tensor (since its rank is $D/2$), yielding relations familiar from Maxwell's theory, though they involve objects of other ranks. These two distinctive properties (linearity and intrinsically relativistic one) seem to be of a great physical importance, which single out these fields from many others (but probably not from the Machian $r = D-1$ fields filling the lower nontrivial diagonal in the Table 1). The Machian intrinsically relativistic fields correspond to $k = 1/2$, hence to $\Lambda = 0$; their components are in fact arbitrary functions of spacetime coordinates, thus such fields differ quite radically from all other physical fields.

In particular, these results yield a

Theorem: (Generalized) electromagnetic fields exist only in even D spacetime dimensions, then being ($r = D/2 - 1$)-form fields. They possess all essential properties of the 3+1 Maxwell fields (are linear, intrinsically relativistic, and subject to the D-dimensional dual conjugation relations).

All other fields in the Table 1 seem to be of less general importance; for example, the ($r = D - 2$)-field models perfect fluids in the respective spacetime, and its Lagrangian needs to be chosen as such a function of the field invariant which yields the desired equation of state.

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